Real Numbers

IIT JAM MS: Real Analysis

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Real Numbers

- For IIT JAM MS, we don't need to go too deep into the workings of real numbers or how to construct them, though these might be useful for understanding the concepts a bit more deeply.
- For this course, we'll assume that $\mathbb{R}$ is a **Complete Ordered Field.**
- $(\mathbb{R}, +, \times)$ is an ordered field.

Ordered Field

The definition of a field is useless for JAM or ISI prep, so think of this as the same as the real numbers you are familiar with.

$+$ is the standard addition, and $\times$ is the standard multiplication.

Ordered

We can compare elements:

$$x > y, \quad x < y, \quad x = y,$$

and this comparison mingles properly with standard addition and multiplication.

Completeness

Completeness is the property that makes $\mathbb{R}$ unique.

The common definition that is used to understand $\mathbb{R}$ is

$$\mathbb{R} = \text{Irrational} \cup \text{Rational}.$$

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Question

What property of $\mathbb{R}$ is not in $\mathbb{Q}$?
How do we differentiate between $\mathbb{Q}$ and $\mathbb{R}$?

Completeness

Consider the set

$$S = \{x : x^2 < 2\}.$$

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Does the set have a maximum element?

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$$\boxed{\text{No.}}$$

But there is a sense in which you want to say that this set has a maximum, that is, $\sqrt{2}$.

Definitions that'll help us formalize this notion of "maximum":

Upper Bound and Least Upper Bound or Supremum

Definition

For every $a \in A$, if $c \geq a$, then c is an **upper bound** of A.

Definition

s is said to be the **supremum** (least upper bound) of A if

1. s is an upper bound of A, and
2. for any other upper bound c of A, $s \leq c$.

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Remark

For now, let's assume that the supremum of S is $\sqrt{2}$. $x^2 = 2$ doesn't have a solution in $\mathbb{Q}$. So, S doesn't have a supremum in $\mathbb{Q}$.

Axiom of Completeness

Also known as **Completeness Axiom** or **Least Upper Bound Property**.

This is an axiom, assumed to be true about $\mathbb{R}$.

Axiom of Completeness

Every subset of $\mathbb{R}$ with an upper bound has a supremum.

Observation

This is the important axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Lower Bound and Greatest Lower Bound or Infimum

Definition

For every $a \in A$, if $c \leq a$, then c is a **lower bound** of A.

Definition

s is said to be the **infimum** (greatest lower bound) of A if

1. s is a lower bound of A, and
2. for any other lower bound c of A, $s \geq c$.

Remark

The **Axiom of Completeness** implies that if A has a lower bound in $\mathbb{R}$, then A has an infimum.

A Few Important Results and Definitions

Archimedean Property

If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $nx > y$.

Rationals are Dense in Reals

If $x, y \in \mathbb{R}$ and $x < y$, then there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x < \dfrac{m}{n} < y$.

ε-Neighbourhood of a

$$V_\varepsilon(a) := \{x \in \mathbb{R} : a - \varepsilon < x < a + \varepsilon\}.$$

Class Problems

1. For $A = (0, 1)$, find all upper bounds you can think of and find $\sup A$.

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3. Consider $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Find $\sup A$ and $\inf A$.

Practice Problems

1. Let $A = \{x \in \mathbb{R} : x^2 < 9\}$. Find $\sup A$ and $\inf A$. Does A have a maximum or minimum?

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1. Let $A = \{x \in \mathbb{R} : x^2 < 9\}$. Find $\sup A$ and $\inf A$. Does A have a maximum or minimum?

2. Let $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Find $\sup A$. Does A have a maximum?

JAM Style Problem

Problem

Let $A = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$. Then which of the following is true?

(A) $\sup A = 1$ and $\max A = 1$

(B) $\sup A = 1$ and A has no maximum

(C) $\sup A < 1$

(D) A is not bounded above

JAM Style Problem

Problem

Let $A = (2, 7]$. Find

$$\sup A, \qquad \inf A, \qquad \max A, \qquad \min A.$$

JAM Style Problem

Problem

Consider $S = \{q \in \mathbb{Q} : q^2 < 2\}$. Which of the following is true?

(A) S has a supremum in $\mathbb{Q}$.

(B) $\sup S = 2$.

(C) S is bounded above in $\mathbb{Q}$ but has no supremum in $\mathbb{Q}$.

(D) S is not bounded above.